BUCKLING OF A VISCOELASTIC BAR

A. M. Lokoshchenko and S. A. Shesterikov

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The behavior of a bar with pin-jointed fixing, having an initial deflection, is investigated when a compressive force remaining constant with time is applied fairly rapidly to the ends of the bar. It is assumed that the material of the bar is such that at the instant of loading the entire bar is in an elastic state. Subsequently, the strain of steady state creep is added to the elastic strains, for stress that exceed

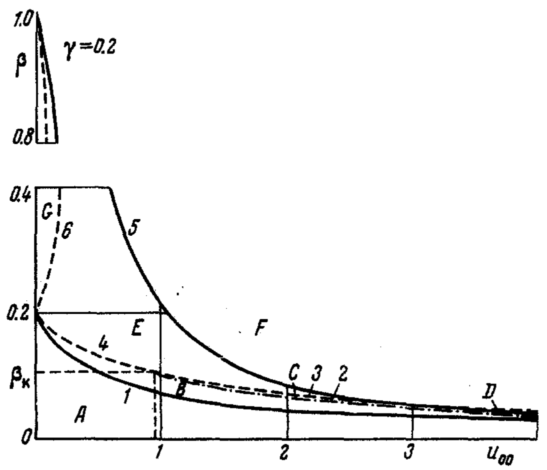

Fig. 1

a certain value. Such a scheme is well confirmed experimentally, for example, in the tests by Wood, Williams, Hodge and Ogden on copper-beryllium wire at slightly elevated temperatures and high stresses [1]. The investigation carried out here is directly related to the solution of the buckling problem of a viscoelastic ideally-plastic bar [2].

§1. We investigate the buckling process under creep conditions of a bar of length l, with pin-jointed fixing; the bar has an initial distortion and is compressed by a constant force P. To simplify the calculations, we assume that the bar has a rectangular cross section of the width b and height 2h. The deflection of the bar is approximated by a single sinusoidal half-wave, and the equation of equilibrium is satisfied at the center point. We assume that the stresses σ_1 and ε_1 are connected as follows:

$$E\varepsilon_1 = \sigma_1 \qquad (|\sigma_1| \leqslant \sigma_T),$$

$$E\frac{d\varepsilon_1}{dt'} = \frac{d\sigma_1}{dt'} + BE(|\sigma_1| - \sigma_T)\,\mathrm{sign}\,\sigma_1 \qquad (|\sigma_1| > \sigma_T). \tag{1.1}$$

Here E is Young's modulus, σ_T is the static yield point and B is the creep characteristic. The compressive stresses and strains are taken as positive. If the application of the forces is such that in certain fibers $|\sigma_1| > \sigma_T$, then with time the stresses can be redistributed over the cross section of the bar. At the same time the boundary (or two boundaries) between the elastic and viscoelastic zones is displaced.

The equations of equilibrium for the bar have the form

$$P = \int_S \sigma_1\, dS, \qquad -P(a + a_{00}) = \int_S \sigma_1 z_1\, dz_1 ,$$

where S is the cross-sectional area, z_1 is the coordinate, measured from the neutral axis, along the cross section S in the plane of flexure, a_{00} is initial flexure in the middle of the bar, and $a(t)$ is the increment of flexure. We adopt the hypothesis of plane sections:

$$\varepsilon_1 = \varepsilon_{10} - \frac{\rho}{h^2} a z_1, \qquad \rho = \frac{\pi^2 h^2}{l^2},$$

where ε_{10} is the strain of the neutral axis. We introduce the dimensionless parameters

$$\sigma = \frac{3\sigma_1}{\rho E}, \quad \gamma = \frac{3\sigma_T}{\rho E}, \quad \beta = \frac{P}{P_\text{э}}, \quad z = \frac{z_1}{h},$$

$$u = \frac{a}{h}, \quad \varepsilon = \frac{3}{\rho}\varepsilon_1, \quad t' = \frac{t}{B} \quad \left(P_\text{э} = \frac{2\pi^2 Ebh^3}{3l^2}\right).$$

Then the equations of equilibrium and the hypothesis of plane sections assume the form

$$2\beta = \int_{-1}^{1} \sigma\, dz, \qquad -2\beta(u + u_{00}) = \int_{-1}^{1} \sigma z\, dz . \tag{1.2}$$

§ 2. Let us investigate the elastic-ideally-plastic state of a bar in compression. We consider a bar made of elastic-ideally-plastic material (with a static yield point) compressed by the force 2β. The exact solution of this problem is given in [3]; here we use an approximate solution in a form which is convenient for the investigation. First we consider the case where only the plastic zone is present in the cross section. Then

$$\sigma = \gamma \qquad (-1 \leqslant z \leqslant b_1),$$

$$\sigma = \varepsilon_0 - 3uz \qquad (b_1 \leqslant z \leqslant 1). \tag{2.1}$$

Substituting Eqs. (2.1) into Eqs. (1.2), we obtain

$$u_{00} = \left[\frac{(b_1+2)}{3\beta} - \frac{4}{3(1-b_1)^2}\right](\gamma - \beta). \tag{2.2}$$

Putting $b_1 = -1$ we obtain the following relation between the parameters β, γ and the initial deflection, u_{00}, corresponding to the instant when plasticity appears on the concave side of the bar

$$u_{00} = \frac{(1-\beta)(\gamma-\beta)}{3\beta}. \tag{2.3}$$

Subsequently, as β increases for $u_{00} < \beta_k^{1/3}$, where β_k is determined from the condition $(1 - \beta_k^{1/3})\gamma = \beta_k$, a failure corresponding to the condition $\partial\beta / \partial u = 0$ takes place. Hence we obtain

$$u_{00} = \frac{1}{2}\left(\frac{\gamma}{\beta} - \frac{\beta}{\gamma} - \frac{\gamma}{\beta^{1/3}}\right) \tag{2.4}$$

We note that β_k is the boundary between two regions; in the first region the limiting state sets in if there is a single plastic region in the cross section; in the second it occurs for two regions. In the case where $u_{00} > \beta_k^{1/3}$, for β satisfying the relation

$$u_{00} = \frac{\gamma}{3\beta} + \frac{1}{3} - \frac{2\beta}{3\gamma} - \frac{\gamma^2}{3(\gamma-\beta)} \tag{2.5}$$

plasticity appears on the convex side of the bar. As β increases further, two plastic regions divided by an elastic core appear in the cross section. For loads larger than those given by the relation (2.5), the elastic core in the bar is encircled from both sides by the plastic zones. In this case

$$\sigma = \gamma \quad (-1 \leqslant z \leqslant b_1),$$

$$\sigma = \varepsilon_0 - 3uz \quad (b_1 \leqslant z \leqslant b_2)$$

$$\sigma = -\gamma \quad (b_2 \leqslant z \leqslant 1). \tag{2.6}$$

Substituting Eqs. (2.6) into Eqs. (1.2), we obtain

$$\frac{\gamma^3}{27u^2} + \frac{\beta^2}{\gamma} - \gamma + 2\beta(u + u_{00}) = 0 . \tag{2.7}$$

Using the condition $\partial\beta / \partial u = 0$ we can find the load for which failure takes place

$$u_{00} = \frac{(\gamma - \beta)(1 - \beta^{1/3})}{\beta}. \qquad (2.8)$$

§3. Let us consider the buckling process of a bar when there is a single viscous region in the cross section.

Before going to investigate this case we must point out that, within the framework of the hypotheses adopted about the properties of the material, a certain region of loads exists for any u_{00} such that in the initial (elastic) stress distribution $|\sigma| \leqslant \gamma$ holds everywhere. Consequently, zones of viscous flow do not appear and the deflections do not vary with time for this load.

We consider the case where this limit is passed. We then assume that the stress distribution in the cross section, for t = 0, satisfies the inequality $\sigma \geqslant \gamma$ for $-1 \leqslant z \leqslant c_0$ and $|\sigma| \leqslant \gamma$ for $c_0 \leqslant z \leqslant 1$. When $-1 \leqslant z \leqslant c_0$ the relation of viscoelasticity is satisfied for all $t \geqslant 0$. In the second zone $c_0 \leqslant z \leqslant c$, where c is the moving boundary between the elastic and viscoelastic zones and $\sigma(c) = \gamma$, creep strains occur for $t > \tau(z)$, where $\tau(z)$ is the instant when c(t) passes through the point z. In the third zone $c \leqslant z \leqslant 1$ only elastic strains occur for $t \geq 0$.

We rewrite the differential relation $\varepsilon^{\cdot} = \sigma^{\cdot} + \sigma - \gamma$ in integral form (a derivative with respect to t is denoted by a dot). We obtain

$$\sigma = \varepsilon - e^{-t}\int_0^t \varepsilon e^t\,dt + \gamma(1 - e^{-t}) \qquad (-1 \leqslant z \leqslant c_0)$$

$$\sigma = \varepsilon - e^{-t}\int_\tau^t \varepsilon e^t\,dt + \gamma(1 - e^{\tau - t}) \qquad (c_0 \leqslant z \leqslant c)$$

$$\sigma = \varepsilon \qquad (c \leqslant z \leqslant 1). \qquad (3.1)$$

From the condition $\sigma(c) = \gamma$ we have $\varepsilon_0 = \gamma + 3uc$. To determine the functions c and u we insert the expression (3.1) into the equation of equilibrium (1.2). To transform the integral relations into

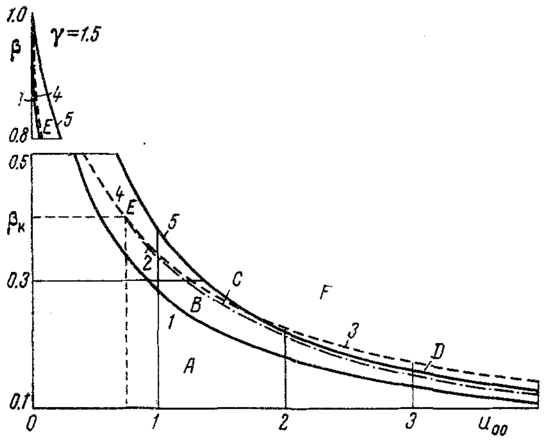

Fig. 2

differential ones we multiply them by e^t, differentiate and then divide them by e^t. As a result, we obtain the system of two first order differential equations:

$$c^{\cdot} = \frac{(1-c)^2}{4} - \left[\frac{(1+c)^2(2-c)}{4(1-\beta)} - 1\right]c - \frac{(c - c_0)\beta u_{00}}{(1-\beta)u},$$

$$c_0 = -\frac{(\gamma - \beta)(1-\beta)}{3\beta u_{00}},$$

$$u^{\cdot} = \left[\frac{(1+c)^2(2-c)}{4(1-\beta)} - 1\right]u + \frac{\beta u_{00}}{(1-\beta)}, \quad u_0 = \frac{\beta u_{00}}{(1-\beta)}. \quad (3.2)$$

The subsequent analysis of the qualitative features of the behavior of the bar in question (for a constant β) is carried out with the use of a phase diagram $\beta \sim u_{00}$. The phase diagram has two different basic forms. Fig. 1 corresponds to the case $\gamma < 1$; Fig. 2 corresponds to the case $\gamma > 1$. The curve 1 in both figures is given by the relation (2.3). It cuts off the region A containing no viscous flow. Consequently, here the bar behaves as ideally elastic.

The curve 2 before contacting the curves 3 and 4 is given by Eq. (2.5). It is easy to show that points of the curve 2 correspond to a state of the bar for which $\sigma(1) \to -\gamma$ for $t \to \infty$. The curve 3 is determined by the relation (2.4).

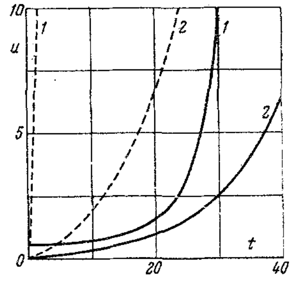

Fig. 3

The relations (3.2) are always operative in the region B bounded by these curves. The stress diagram for the region B, for any ti, is such that $\sigma \geqslant -\gamma$ everywhere. In this case we have the limiting state for $t \to \infty$; it is characterized by the condition $u^{\cdot} = c^{\cdot} = 0$.

Using this condition, from (3.2) we find the equation for the limiting value c_∞:

$$3\beta u_{00}(1 - c_\infty)^2 =$$

$$= [4(1-\beta) - (1 + c_\infty)^2(2 - c_\infty)](\gamma - \beta). \qquad (3.3)$$

It is easy to show that Eq. (3.3) coincides with Eq. (2.2) when c_∞ is replaced by b_1. The value u_∞ can be found from the second Eq. (3.2), if we put $u^{\cdot} = 0$. Consequently, the bar is asymptotically stable in the region B. At the same time a certain viscous flow takes place, transforming the bar from the purely elastic state into the limiting elastic-ideally-plastic state after infinitely long time. The region B fully occupies the part of the phase diagram where the relations (3.2) are true for any t.

Points lying above the curves 2 and 3 are characterized by the fact that at a certain instant $t = t_1$ (t_1 can be zero) a viscous region appears on the convex side of the bar. Points lying above the curve 5 correspond to the case $t_1 = 0$. The relations (3.2) are true for points of the regions E and C for $t < t_1$. The value of the time t_1 is determined from the condition $\sigma(1, t_1) = -\gamma$.

§4. Let us consider the second basic case, when viscous regions with stresses of different signs develop on both sides of the bar, with the boundaries c and d on the concave and convex side respectively. Transforming Eqs. (1.1) into integral form, we have

$$\sigma = \varepsilon - e^{-t}\int_{t_1}^t \varepsilon e^t\,dt + \gamma(1 - e^{t_1 - t}) \qquad (-1 \leqslant z \leqslant c_0),$$

$$\sigma = \varepsilon - e^{-t}\int_{\tau_1(z)}^t \varepsilon e^t\,dt + \gamma(1 - e^{\tau_1 - t}) \qquad (c_0 \leqslant z \leqslant c),$$

$$\sigma = \varepsilon \qquad (c \leqslant z \leqslant d),$$

$$\sigma = \varepsilon - e^{-t}\int_{\tau_2(z)}^t \varepsilon e^t\,dt - \gamma(1 - e^{\tau_2 - t}) \qquad (d \leqslant z \leqslant d_0),$$

$$\sigma = \varepsilon - e^{-t}\int_{t_1}^t \varepsilon e^t\,dt - \gamma(1 - e^{t_1 - t}) \qquad (d_0 \leqslant z \leqslant 1). \quad (4.1)$$

In addition, for c and d we can easily obtain the relation

$$3u(d - c) = 2\gamma. \qquad (4.2)$$

For the determination of the functions u, c and d we have Eqs. (4.1), (4.2) and the equation of equilibrium (1.2). These equations are reduced to the two equations for c and d:

$$c^{\cdot} = -\frac{\beta c}{(1-\beta)} + \frac{(d-c)}{4(1-\beta)}\left[\left(\frac{2\beta}{\gamma} - d - c\right) \times \right.$$
$$\left. \times (1-\beta) - \frac{6\beta u_{00} c}{\gamma} + 3c - cd^2 - c^2 d - c^3\right],$$
$$d^{\cdot} = -\frac{\beta d}{(1-\beta)} + \frac{(d-c)}{4(1-\beta)}\left[\left(\frac{2\beta}{\gamma} - d - c\right) \times \right.$$
$$\left. \times (1-\beta) - \frac{6\beta u_{00} d}{\gamma} + 3d - cd^2 - c^2 d - d^3\right]. \qquad (4.3)$$

The initial conditions for the system (4.3) can have two forms. If the point in the phase diagram lies in the regions F and D, then $t_1 = 0$ and

$$c(0) = \frac{(1-\beta)(\beta-\gamma)}{3\beta u_{00}}, \qquad d(0) = \frac{(1-\beta)(\beta+\gamma)}{3\beta u_{00}}. \qquad (4.4)$$

If the point is located in the regions E and C, the Eqs. (4.3) hold for $t > t_1$ and the initial values for c are determined from the

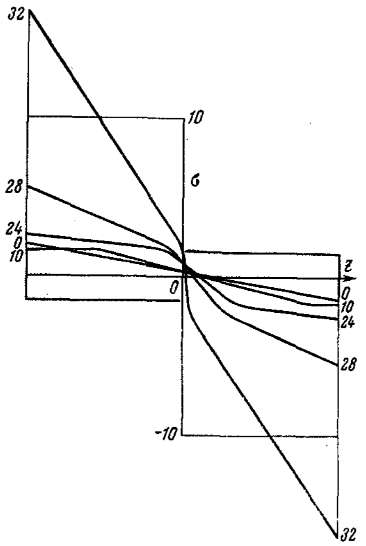

Fig. 4

solution of the system (3.2) for $t = t_1$, whereas $d(t) = 1$ is determined by the definition.

Let us consider the regions C and D in the phase diagram. Both these regions are characterized by the fact that for u_{00} and β belonging to these regions we have a certain finite limiting value for c, d and u for $t \to \infty$. This arises from the existence of the limiting values (4.3) for the condition $c^{\cdot} = d^{\cdot} = 0$ and for $c \neq d$. It is easy to show that the limiting values for c, d and u coincide with the corresponding parameters when the elastic-ideally-plastic problem is solved. Consequently, the regions C and D are also regions of asymptotic stability. The curves 3 and 4 (the curve 4 is given by the relation (2.8)) separate the regions of stability from the regions where the deflection increases without bounds with time (for $t \to \infty$). This conclusion follows from the analysis of Eqs. (4.3) with Eqs. (4.2) taken into account. We also note that this boundary coincides with the boundary corresponding to the condition $\partial\beta / \partial u = 0$ in the elastic-ideally-plastic scheme of behavior of the material.

In the region E we have $t_1 > 0$, and in the region F we have $t_1 = 0$. In these regions we have boundless increase of the deflection u for $t \to \infty$. For sufficiently long times, u increases with time exponentially as for a linearly viscoelastic bar. Consequently, an actual bar can work in the regions E and F for only a limited time (for example, for given conditions imposed on the deflections). The regions E and F are divided by the curve 5 which is determined from the condition $\sigma(1, 0) = -\gamma$.

In the case when $\gamma < 1$ we have an additional region G in the phase diagram. It is characterized by the fact that at the initial instant $\sigma > \gamma$ everywhere, and the viscous flow envelops the entire cross section of the bar. Using the condition $\sigma(1) = \gamma$ in the elastic solution, we obtain the equation of the curve 6:

$$u_{00} = \frac{(1-\beta)(\beta-\gamma)}{3\beta}. \qquad (4.5)$$

For a bar with u_{00} and β in the region G, the relation between u and t for small t has the form

$$u = \frac{u_{00}}{(1-\beta)}\left[\exp\frac{\beta t}{(1-\beta)} - 1 + \beta\right]. \qquad (4.6)$$

The relation (4.6) is operative for $t \leq t_2$; here t_2 is determined from the condition $\sigma(1, t_2) = \gamma$. When $t > t_2$, an elastic zone appears on the convex side of the bar and an equation of the type (3.2) becomes operative. When $t = t_1$, the condition $\sigma(1, t_1) = -\gamma$ is satisfied. When $t > t_1$ a system of the type (4.3) becomes decisive, and for $t \to \infty$, for any $u_{00} \neq 0$ of this region, $u \to \infty$. The systems of nonlinear equations (3.2) and (4.3) can be solved only numerically. These systems are solved on a digital computer Nairi for a number of more characteristic values of the parameters γ, β and u_{00}. In Fig. 3 continuous lines represent the calculated curves u(t) for the following cases:

index 1 denotes the curve for $\gamma = 1.5, \beta = 0.25, u_{00} = 1.8$;
index 2 denotes the curve for $\gamma = 0.2, \beta = 0.1, u_{00} = 1$.

For comparison purposes we have shown on the same graph, by dotted lines, the curves u(t) for the condition $\gamma = 0$ (purely viscoelastic bar). It is seen that even in the case where the limiting state is absent, but the point in the phase diagram is close to the region of asymptotic stability, the calculated curves sharply diverge from the curves for the purely viscoelastic bar. At the same time (curve 1, for example), for the initial portion of t, the deflection u increases very slowly (nearly linearly with time). This is in contrast to the viscoelastic bar.

The variation with time of the stress diagram over the cross section of the bar is of interest. As one of the more interesting cases, in Fig. 4 we have shown the dynamics of the variation of the σ diagram for the case $\gamma = 1.5$, $\beta = 0.25$, $u_{00} = 1.8$ (the numbers on the right and left denote the corresponding dimensionless time). From the graph (also obtained numerically on the computer Nairi) it follows that the initial linear (elastic for $t = 0$) stress distribution for small t tends to the elastic-ideally-plastic case. The "relaxation" effects prevail during this period.

The bending moment increases with the subsequent growth of the deflections, and the stress distribution again tends to be linear over the cross section, with the exception of the breaks on the boundaries between the elastic and viscoelastic zones. Here the boundary c crosses 0 and for the subsequent increase of the deflection will tend to 0 on the right. The boundary d approaches 0 monotonically on the right (remaining always more than c on the right). Having considered a large number of cases, we conclude that when asymptotic stability is absent, $c \to +0$ regardless of the sign of c(0).

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